

BOUNDARY EXPANSIONS OF COMPLETE CONFORMAL METRICS WITH NEGATIVE RICCI CURVATURES

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ABSTRACT. We study the boundary behaviors of a complete conformal metric which solves the σ_k -Ricci problem on the interior of a manifold with boundary. We establish asymptotic expansions and also C^1 and C^2 estimates for this metric multiplied by the square of the distance in a small neighborhood of the boundary.

1. INTRODUCTION

Let $(M, \partial M, g)$ be a smooth Riemannian manifold with boundary and $1 \leq k \leq n$. We consider the following problem:

$$(1.1) \quad \sigma_k[-g^{-1}Ric(e^{2u}g)] = (n-1)^k C_n^k e^{2ku} \quad \text{in } M \setminus \partial M,$$

$$(1.2) \quad u = \infty \quad \text{on } \partial M,$$

where $C_n^k = \binom{n}{k}$, $Ric(e^{2u}g)$ is the Ricci curvature of the conformal metric $e^{2u}g$, and $\sigma_k(A)$ is the k -th elementary symmetric polynomial in the eigenvalues of the symmetric matrix A . Let Γ_k^+ be the connected component of the set $\{\sigma_k > 0\}$ which contains the positive definite cone.

Gursky, Streets and Warren [5] proved that (1.1) and (1.2) admit a unique solution $u \in C^\infty(M \setminus \partial M)$ with an additional requirement that $-Ric(e^{2u}g) \in \Gamma_k^+$. Moreover, $e^{2u}g$ is a complete metric and

$$(1.3) \quad \lim_{x \rightarrow \partial M} [u + \log d] = 0,$$

where d is the distance to ∂M . Refer to Theorem 1.4 in [5]. By comparing (1.1) with the equation in Theorem 1.4 [5], we note that a constant $(n-1)^k C_n^k$ is inserted in the right-hand side of (1.1). With the newly inserted constant factor, the constant term in the expansion (1.3) is zero.

In this paper, we study further expansions of u near the boundary. For brevity, we consider the case that g is the standard Euclidean metric. Assume $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain, for $n \geq 3$. For $u \in C^2(\Omega)$, define a symmetric matrix $A(u)$ by

$$(1.4) \quad A(u) = (n-2)\nabla^2 u + \Delta u I_{n \times n} + (n-2)[|\nabla u|^2 I_{n \times n} - \nabla u \otimes \nabla u],$$

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where $I_{n \times n}$ is the identity $n \times n$ matrix. We are led to the following problem:

$$(1.5) \quad \sigma_k(A(u)) = (n-1)^k C_n^k e^{2ku} \quad \text{in } \Omega,$$

$$(1.6) \quad u = \infty \quad \text{on } \partial\Omega,$$

with the additional requirement that $A(u) \in \Gamma_k^+$.

Set

$$e^{2u} = w^{\frac{4}{n-2}}.$$

For $k = 1$, (1.5) and (1.6) are reduced to the following more familiar form:

$$(1.7) \quad \begin{aligned} \Delta w &= \frac{1}{4} n(n-2) w^{\frac{n+2}{n-2}} \quad \text{in } \Omega, \\ w &= \infty \quad \text{on } \partial\Omega. \end{aligned}$$

Loewner and Nirenberg [7] proved the existence of the unique positive solution of (1.7) and Aviles and McOwen [2] proved the same result for the corresponding equation in general manifolds. Andersson, Chruściel and Friedrich [1] and Mazzeo [8] established the polyhomogeneous expansions for the solutions. Graham [4] studied the renormalized volume expansion. He identified the first two renormalized volume coefficients and the information contained in the anomaly, namely, the difference of the renormalized volumes corresponding to different choices of conformal representatives, and proved the conformal invariance of the energy, the coefficient of the log-term in the volume expansion.

We now present our main results for (1.5) and (1.6). As in (1.3), we denote by d the distance function in Ω to $\partial\Omega$ and set

$$(1.8) \quad D_\delta = \{x \in \Omega \mid d(x) \leq \delta\} \cap \Omega.$$

If $\partial\Omega$ is C^∞ , then d is C^∞ in a sufficiently small neighborhood of $\partial\Omega$. In this paper, we use the principal coordinates in D_δ and denote by (x', d) the points in D_δ , for $\delta > 0$ sufficiently small.

We have the following result for the expansions of $u + \log d$ up to the first log-term with the coefficients in terms of principal curvatures and their derivatives.

Theorem 1.1. *Assume that Ω is a bounded smooth domain in \mathbb{R}^n , for $n \geq 3$, and that u is the solution of (1.5)-(1.6). Then,*

$$(1.9) \quad |u + \log d - c_1 d - \cdots - c_{n-1} d^{n-1} - c_{n,1} d^n \log d| \leq C d^n \quad \text{in } D_{\delta_2},$$

where C and δ_2 are positive constants depending only on Ω , n and k , and c_1, \dots, c_{n-1} and $c_{n,1}$ are smooth functions on $\partial\Omega$.

We note that c_1, \dots, c_{n-1} and $c_{n,1}$ will be given by (2.6), (2.7), and (2.8).

We point out that Theorem 1.1 holds for solutions of (1.1) and (1.2), not just for those of (1.5) and (1.6). To emphasize the dependence of solutions on k , we denote by u_k the solution of (1.5) and (1.6) and write its expansion as

$$(1.10) \quad w_k = -\log d + c_1^k d + c_2^k d^2 + \cdots + c_{n-1}^k d^{n-1} + c_{n,1}^k d^n \log d + O(d^n).$$

Denote by g_0 the Euclidean metric. As mentioned earlier, u_k is the solution of (1.1) and (1.2) when $g = g_0$. Consider a conformal metric $g = e^{2\rho}g_0$ and the corresponding solution \tilde{w}_k of (1.1) and (1.2). Then, \tilde{w}_k also has an expansion in the form

$$(1.11) \quad \tilde{w}_k = -\log d + \tilde{c}_1^k d + \tilde{c}_2^k d^2 + \cdots + \tilde{c}_{n-1}^k d^{n-1} + \tilde{c}_{n,1}^k d^n \log d + O(d^n),$$

In fact, we can take $\tilde{w}_k + \rho$ as a function and then apply the uniqueness result for the Euclidean metric g_0 . A direct consequence of Theorem 1.1 is the following result, which we can compare with results in [4].

Proposition 1.2. *Let c_j^k , $j = 1, \dots, n-1$, be the terms as in (1.10). Then,*

$$(1.12) \quad c_j^k - c_j^1 \text{ is conformally invariant;}$$

namely,

$$c_j^k - c_j^1 = \tilde{c}_j^k - \tilde{c}_j^1,$$

where \tilde{c}_j^k is given in (1.11), $j = 1, \dots, n-1$.

Next, we derive the C^1 and C^2 estimates for $u + \log d$.

Theorem 1.3. *Assume that Ω is a bounded smooth domain in \mathbb{R}^n , for $n \geq 3$, and that u is the solution of (1.5)-(1.6). Then,*

$$|\nabla(u + \log d - c_1 d)| \leq C d^\alpha \quad \text{in } D_{\delta_3},$$

where C and δ_3 are positive constants depending only on Ω , n and k , c_1 is the function in (2.6), $\alpha = 1/2$ when $n = 3$ and $\alpha = 1$ when $n \geq 4$.

Theorem 1.4. *Assume that Ω is a bounded smooth domain in \mathbb{R}^n , for $n \geq 8$, and that u is the solution of (1.5)-(1.6). Then,*

$$|\nabla^2(u + \log d - c_1 d)| \leq C \quad \text{in } D_{\delta_4},$$

where C and δ_4 are positive constants depending only on Ω , n and k , and c_1 is the function in (2.6).

The paper is organized as follows. In Section 2, we prove the boundary expansion of $u + \log d$ and the conformal invariance of the difference of corresponding coefficients in the expansions. In Section 3 and Section 4, we derive the C^1 and C^2 estimates for $u + \log d$, respectively.

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2. BOUNDARY EXPANSIONS

Consider the operators

$$(2.1) \quad F(u) = \sigma_k(\lambda(A(u))) - (n-1)^k C_n^k e^{2ku},$$

and

$$(2.2) \quad \tilde{F}(u) = F(u)d^{2k}.$$

By [5], there exists a unique solution $u \in C^\infty(\Omega)$ of (1.5)-(1.6). Then, $F(u) = 0$ in Ω .

Our goal in this section is to derive boundary expansion for u involving all local terms by the maximum principle in a sufficiently small neighborhood of $\partial\Omega$.

Denote by (x', d) the principle coordinates near boundary and by $\kappa_1, \dots, \kappa_{n-1}$ the principal curvatures of $\partial\Omega$, with respect to the interior unit normal vectors. We set

$$H_{\partial\Omega} = \kappa_1 + \dots + \kappa_{n-1}.$$

In the following, we calculate the operator F on polynomials of d and always use C for a positive constant depending only on Ω , n and k . We set

$$(2.3) \quad v = -\log d + c_0 + c_1 d + c_2 d^2 + \dots + c_{n,1} d^n \log d,$$

where c_i $i = 1, \dots, n-1$, and $c_{n,1}$ are functions of x' to be determined.

Lemma 2.1. *Let Ω be a bounded smooth domain in \mathbb{R}^n , for $n \geq 3$. Then, there exist functions c_1, \dots, c_{n-1} , and $c_{n,1}$ defined on $\partial\Omega$ such that, for v defined in (2.3),*

$$(2.4) \quad \tilde{F}(v) = O(d^{n+\tau}),$$

where τ is an arbitrarily given positive constant in $(0, 1)$.

Proof. For v as in (2.3), set

$$\tilde{A}_{ij} = \tilde{A}_{ij}(v) = d^2 A_{ij}(v).$$

A straightforward calculation yields

$$(2.5) \quad \begin{aligned} \tilde{A}_{aa} &= (n-1) + (n-2)\kappa_a d + H_{\partial\Omega} d - 2(n-2)c_1 d \\ &\quad + \text{polynomial of } d \text{ with power higher than } 2, \\ \tilde{A}_{an} &= \tilde{A}_{ab} = \text{polynomial of } d \text{ with power higher than } 2, \\ \tilde{A}_{nn} &= (n-1) + H_{\partial\Omega} d + \text{polynomial of } d \text{ with power higher than } 2. \end{aligned}$$

We now substitute v in

$$\tilde{F}(v) = \sigma_k(\lambda(\tilde{A}(v))) - d^{2k}(n-1)^k C_n^k e^{2kv},$$

and arrange $\tilde{F}(v)$ in an ascending order of d . By requiring the constant term and the coefficient of d to be zero in $\tilde{F}(v)$, we have

$$c_0 = 0,$$

and

$$(2.6) \quad c_1 = \frac{1}{2(n-1)} H_{\partial\Omega}.$$

For $i = 2, \dots, n-1$, by requiring the coefficient of d^i , $i = 2, \dots, n-1$, to be zero in $\tilde{F}(v)$ successively, we have

$$(2.7) \quad c_i = \frac{1}{(i-n)(i+1)} G_i(c_1, c_2, \dots, c_{i-1}),$$

where G_i is a smooth function in c_1, \dots, c_{i-1} and their derivatives. We point out that the coefficient of $d^n \log d$ in $\tilde{F}(v)$ equals 0. By requiring the coefficient of d^n to be zero in $\tilde{F}(v)$, we have

$$(2.8) \quad c_{n,1} = \frac{1}{(n-1)(n+1)} G_n(c_1, c_2, \dots, c_{n-1}),$$

where G_i is a smooth function in c_1, \dots, c_{n-1} and their derivatives. Then we obtain the desired result. \square

The functions c_1, \dots, c_{n-1} and $c_{n,1}$ defined in (2.6), (2.7) and (2.8) are functions on $\partial\Omega$. They are the coefficients of the so-called local terms, since they can be expressed explicitly in terms of principal curvatures and their derivatives. For a demonstration, we calculate c_2 .

Proposition 2.2. *The function c_2 in (2.7) has the following expression:*

$$(2.9) \quad c_2 = \frac{n}{6(n-2)} \left\{ \left(\frac{-3n+2}{4n(n-1)^2} - \frac{n^3-3n-n^2+4}{2n(n-1)^4} \right) H_{\partial\Omega}^2 + \left(\frac{2}{n} + \frac{(n-2)^2}{2n(n-1)^3} \right) |\Pi|^2 \right\} \\ + k \left(-\frac{(n-2)}{12(n-1)^3} \right) |\overset{\circ}{\Pi}|^2,$$

where Π is the second fundamental form and $\overset{\circ}{\Pi}$ is the trace-free second fundamental form, i.e.,

$$\overset{\circ}{\Pi} = \Pi - \frac{1}{n} Hg.$$

We note that c_2 can be expressed as the sum of two parts, the first part independent of k and the second part a conformal invariant multiplied by k .

Proof. By (2.7) and (2.6), we have

$$\frac{6}{n}(n-2)c_2 = I + \widehat{I},$$

where

$$I = -\frac{1}{(n-1)^3 n} \left\{ \sum_{a < b} \left(\frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_a \right) \left(\frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_b \right) + (n-1) H_{\partial\Omega}^2 \right\} + \frac{-3n+2}{4n(n-1)^2} H_{\partial\Omega}^2 + \frac{2}{n} |\Pi|^2,$$

$$\hat{I} = \frac{k}{(n-1)^3 n} \left\{ \sum_{a < b} \left(\frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_a \right) \left(\frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_b \right) + (n-1) H_{\partial\Omega}^2 + \left((n-1) - \frac{(n-1)n}{2} \right) H_{\partial\Omega}^2 \right\}.$$

Set

$$D = \sum_{a < b} \left(\frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_a \right) \left(\frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_b \right) + (n-1) H_{\partial\Omega}^2.$$

Then,

$$D = \frac{n^3 - 3n - n^2 + 4}{2(n-1)} H_{\partial\Omega}^2 - \frac{(n-2)^2}{2} |\Pi|^2,$$

$$|\overset{\circ}{\Pi}|^2 = \sum_a \left(\kappa_a - \frac{1}{n-1} H_{\partial\Omega} \right)^2 = |\Pi|^2 - \frac{1}{n-1} H_{\partial\Omega}^2,$$

and

$$D - \frac{(n-1)n}{2} H_{\partial\Omega}^2 = -\frac{(n-2)^2}{2} \left(|\Pi|^2 - \frac{1}{n-1} H_{\partial\Omega}^2 \right) = -\frac{(n-2)^2}{2} |\overset{\circ}{\Pi}|^2.$$

Hence, we have (2.9). \square

Before deriving boundary expansion for u , we show a version of the maximum principle, which will be of use to us.

Theorem 2.3. *Let Ω be a bounded smooth domain in \mathbb{R}^n , for $n \geq 3$. Suppose u and v are smooth sub- and super-solutions, respectively, to (1.5)-(1.6) in Ω and $A(v) \in \Gamma_k^+$. If $\lim_{x \rightarrow \partial\Omega} (u - v) \leq 0$, then $u \leq v$ in Ω .*

Proof. Suppose that $u > v$ somewhere in Ω . Let C be the maximum of $u - v$, which is attained at some point $x_0 \in \Omega$. Then, $w = u - C$ is a strict sub-solution to (2.1). Hence at the point x_0 , we have $w(x_0) = v(x_0)$ and $F(w)(x_0) > F(v)(x_0)$. Then,

$$\sigma_k(\lambda(A(w)))(x_0) > \sigma_k(\lambda(A(v)))(x_0).$$

However, $v \geq w$ near x_0 . Therefore, we have $dw(x_0) = dv(x_0)$ and $(v - w)_{ij}(x_0) \geq 0$, and hence $A(w)(x_0) \leq A(v)(x_0)$. We use Lemma 3.1 in [6] and then obtain

$$\sigma_k(\lambda(A(w)))(x_0) \leq \sigma_k(\lambda(A(v)))(x_0).$$

This leads to a contradiction. \square

According to Theorem 1.4 in [5], the solution u to (1.5)-(1.6) has the decay estimate (1.3). Now, we prove that the decay rate is actually $O(d)$.

Lemma 2.4. *Assume that Ω is a bounded smooth domain in \mathbb{R}^n , for $n \geq 3$ and that u is the solution of (1.5)-(1.6). Then,*

$$(2.10) \quad |u + \log d| \leq Cd \quad \text{in } D_{\delta_1},$$

where C and δ_1 are positive constants depending only on Ω , n and k .

Proof. By (1.3), we can take a small positive constant ε to be determined and then a small enough positive constant δ_0 depending on ε such that

$$(2.11) \quad |u + \log d| \leq \varepsilon \quad \text{in } D_{\delta_0}.$$

Set

$$\phi = -\log d + Cd.$$

Take a small positive constant $\delta_1 < \delta_0$ to be determined and set

$$(2.12) \quad C = \frac{\varepsilon}{\delta_1}.$$

Then by (2.11), (2.12) and (1.3), we have

$$u \leq -\log d + Cd \quad \text{on } \partial D_{\delta_1},$$

and

$$(2.13) \quad Cd \leq \varepsilon \quad \text{in } D_{\delta_1}.$$

We use principle coordinates in D_{δ_1} . By (2.5) and (2.13), we have, when $\varepsilon \ll 1$ and δ_1 are small,

$$\begin{aligned} F(\phi) &= \frac{1}{d^{2k}} \sigma_k(\lambda(d^2 A(\phi))) - d^{2k} (n-1)^k C_n^k e^{2k\phi} \\ &= \frac{1}{d^{2k}} \{ (n-1)^{k-1} C_{n-1}^{k-1} [(2(n-1))H_{\partial\Omega}d - 2(n-2)(n-1)Cd] \\ &\quad - (n-1)^k C_n^k 2kCd + O(\varepsilon Cd) \}. \end{aligned}$$

Hence, when δ_1 and ε are small enough and thus C big enough, we have $F(\phi) < 0$ in D_{δ_1} . Here, by the definition of C in (2.12), we know that the choices of δ_1 and ε are independent. On the other hand, by (2.5) and (2.13), when δ_1 and ε are small enough, we have

$$\sigma_k(\lambda(A(\phi))) = \frac{1}{d^{2k}} (\sigma_k(\lambda(d^2 A(\phi)))) > 0.$$

Obviously, $A(\phi) \in \Gamma_k^+$. Therefore, by using Theorem 2.3, we have $u \leq \phi = -\log d + Cd$ in D_{δ_1} . Similarly, we can prove $u \geq -\log d - Cd$ in D_{δ_1} . \square

Now, we can derive the boundary expansion for u involving all local terms.

Proof of Theorem 1.1. Take δ_2 small to be determined such that $\delta_2 \leq \delta_1$, where δ_1 is as in Lemma 2.4. Consider in D_{δ_2} . For any fixed $\varepsilon \in (0, 1)$, set

$$(2.14) \quad \begin{aligned} A &= 2C\delta_2^{1-n}, \\ q &= n + \varepsilon, \end{aligned}$$

where C is a large enough constant depending on the constant in (2.10) and $\partial\Omega, n, k$. By the definition of A , when δ_1 is small,

$$Ad^n - Ad^q \geq \frac{A}{2}d^n \geq Cd \quad \text{on } \partial D_{\delta_2},$$

and

$$(2.15) \quad Ad^n \leq 2C\delta_2 \quad \text{in } D_{\delta_2}.$$

Hence, for a positive constant $\mu \ll 1$ to be determined, we can choose δ_2 small such that

$$(2.16) \quad Ad^n \leq \mu.$$

Next, set

$$\varphi = Ad^n - Ad^q,$$

and

$$\bar{v} = v + \varphi,$$

$$\underline{v} = v - \varphi.$$

where c_i, \dots, c_{n-1} , and $c_{n,1}$ are the functions on $\partial\Omega$ and v is defined in (2.3). Then by (2.4) and (2.16), a straightforward calculation yields, in D_{δ_2} ,

$$(2.17) \quad \tilde{F}(\bar{v}) = -C_{n-1}^{k-1}(n-1)^k(2\varepsilon)(n+1+\varepsilon)Ad^{n+\varepsilon} + O(\mu Ad^{n+\varepsilon}).$$

Choose δ_2 small enough and thus μ small by (2.15) and A large by (2.14). Then, $\tilde{F}(\bar{v}) < 0$ and therefore $F(\bar{v}) = \frac{1}{d^{2k}}\tilde{F}(\bar{v}) < 0$ in D_{δ_2} . Next by (2.16), we have $A(\bar{v}) \in \Gamma_k^+$, if δ_2 is small. By the maximum principle Theorem 2.3, $u \leq \bar{v}$ in D_{δ_2} . Similarly, we have $u \geq \underline{v}$ in D_{δ_2} . Hence, we have the desired result. \square

Next, we prove the conformal invariance of the difference of the coefficients in expansions as described in Proposition 1.2.

Proof of Proposition 1.2. For $g = e^{2\rho}g_0$, as discussed in Section 1,

$$(2.18) \quad \sigma_k[-g^{-1}e^{-2\tilde{w}_k}\text{Ric}(e^{2\tilde{w}_k}g)] = \beta_{k,n}$$

is equivalent to

$$(2.19) \quad \sigma_k[-g_0^{-1}e^{-2(\rho+\tilde{w}_k)}\text{Ric}(e^{2(\rho+\tilde{w}_k)}g_0)] = \beta_{k,n}.$$

Hence, $w_k = \rho + \tilde{w}_k$. Assume that ρ has the expansion

$$\rho = \rho_0 + \rho_1 d + \dots + \rho_{n-1} d^{n-1} + O(d^n),$$

where ρ_1, ρ_2, \dots are functions on $\partial\Omega$. By the expansion in Theorem 1.1, we have, for $j = 1, \dots, n-1$,

$$\tilde{c}_j^k - \tilde{c}_j^1 = (c_j^k - \rho_j) - (c_j^1 - \rho_j) = c_j^k - c_j^1.$$

This is the desired result. \square

3. THE C^1 -ESTIMATES

In this section, we prove C^1 estimate for $u + \log d$ in a sufficiently small neighborhood of $\partial\Omega$ where u is the solution to (1.5)-(1.6).

Lemma 3.1. *Assume that Ω is a bounded smooth domain in \mathbb{R}^n , for $n \geq 3$, and that u is the solution of (1.5)-(1.6). Then*

$$|\nabla(u + \log d - c_1 d)| \leq C \quad \text{in } D_{\delta_3},$$

where c_1 is the function in (2.6), and C and δ_3 are positive constants depending only on Ω , n and k .

Proof. Take δ_2 as the constant in Theorem 1.1 and $\tilde{c}_1, \psi \in C^\infty(\Omega)$ satisfying

$$\tilde{c}_1 = c_1, \psi = d \quad \text{in } D_{\frac{1}{2}\delta_2},$$

and

$$\psi \geq \frac{1}{2}\delta_2, \quad \text{in } \Omega \setminus D_{\frac{1}{2}\delta_2},$$

where c_1 is the function as given in (2.6). Set

$$w = u + \log \psi - \tilde{c}_1 \psi.$$

We will prove for some $C_0 > 1$,

$$(3.1) \quad \left| \frac{w}{\psi^2} \right| \leq C_0 \quad \text{in } \Omega.$$

First, by Theorem 1.1, we know (3.1) holds in $D_{\delta_2/2}$. Next, take

$$j_1 = -\log(\frac{1}{2}\delta_2) + C\delta_2, \quad j_2 = -\log(\frac{1}{2}\delta_2) - C\delta_2.$$

By Remark 4.10 in [5], for $i = 1, 2$, respectively, we can solve

$$(3.2) \quad F(u_{j_i}) = 0 \quad \text{in } \Omega \setminus D_{\frac{1}{2}\delta_2},$$

$$(3.3) \quad u_{j_i} = j_i \quad \text{on } \partial(\Omega \setminus D_{\frac{1}{2}\delta_2}).$$

By maximum principle and Lemma 2.4, we obtain $u_{j_2} \leq u \leq u_{j_1}$ in $\Omega \setminus D_{\delta_2/2}$. Hence, (3.1) holds in $\Omega \setminus D_{\frac{1}{2}\delta_2}$.

We rewrite the equation (1.5) as

$$(3.4) \quad \sigma_k(\psi^2(\overline{A}(w - \log \psi + \tilde{c}_1 \psi))) = e^{2k\tilde{c}_1\psi} \left(\frac{n-1}{n-2}\right)^k C_n^k e^{2kw} \doteq e^{2k\tilde{c}_1\psi} \beta_{n,k} e^{2kw} \quad \text{in } \Omega,$$

where

$$(3.5) \quad (\overline{A}(u))_{ij} = \partial_{ij}u + \frac{1}{n-2}\Delta u \delta_{ij} + |\nabla u|^2 \delta_{ij} - \partial_i u \partial_j u.$$

We denote the $(k-1)$ -Newton transformation associated with $\psi^2 \overline{A}(w - \log \psi + \tilde{c}_1 \psi)$ as $T_{k-1} \doteq T$, which is positive since $\psi^2 \overline{A} \in \Gamma_k^+$. In particular, if A_j^i are the components of a symmetric matrix A , then the q th Newton transformation associated with A is

$$T_q(A)_j^i = \frac{1}{q!} \delta_{j_1 j_2 \dots j_q j}^{i_1 i_2 \dots i_q i} A_{i_1}^{j_1} \dots A_{i_q}^{j_q}.$$

Here $\delta_{j_1 j_2 \dots j_q j}^{i_1 i_2 \dots i_q i}$ is the generalized Kronecker delta symbol. We frequently use the following properties of $T_{k-1}(A)$:

$$(3.6) \quad \begin{aligned} T_{k-1}(A)_{ij} A_{ij} &= k \sigma_k(A); \\ \text{tr} T_{k-1}(A) &= (n - k + 1) \sigma_{k-1}(A); \\ \partial_m(\sigma_k(A)) &= T_{k-1}(A)_{ij} \partial_m(A_{ij}). \end{aligned}$$

Set

$$Q_{ij} = T_{ij} + \frac{1}{n-2} T_{ll} \delta_{ij}.$$

There is a summation in l . Then, Q_{ij} is positive definite. For the definition and properties of Newton transformation, we can refer to [6]. Set

$$\phi(s) = \frac{1}{p^2 (3C_0)^p} (2C_0 + s)^p,$$

for some p large to be determined and C_0 as in (3.1). Then,

$$\frac{1}{p^2} \geq \phi(s) > 0 \quad \text{for any } s \in [-C_0, C_0].$$

Set

$$h = (1 + \frac{|\nabla w|^2}{2}) e^{\phi(\frac{w}{\psi^2})} \doteq v e^{\phi(\frac{w}{\psi^2})}.$$

We will prove, for some constant C ,

$$|h|_{L^\infty(\Omega)} \leq C.$$

This implies the desired result.

First, for any point $x_0 \in \partial\Omega$, take the principal coordinates (x', d) at x_0 with the unit inner normal vector ν in the x_n -direction. By Theorem 1.1, we know $w \equiv 0$ on $\partial\Omega$ and $w \leq Cd^2$ in D_{δ_2} . Hence, $\nabla_{x'} w \equiv 0$ on $\partial\Omega$ and

$$|\frac{\partial w}{\partial \nu}(x_0)| = |\lim_{d \rightarrow 0} \frac{w(x'_0, d) - 0}{d - 0}| = 0.$$

Hence, $\nabla w(x_0) = 0$, implying $|h(x_0)| \leq C$.

Thus, without loss of generality, we can assume that the maximum of h attains at a point $x_0 \in \Omega$. The proof is inspired by [6]. Assume $|\nabla w(x_0)|$ is sufficiently large. Otherwise the conclusion is immediate. All the calculation below is at the point x_0 . For brevity, we write

$$s = \frac{w}{\psi^2}.$$

Differentiate h twice. Since Q_{ij} is positive definite, we have

$$h_i = 0, \quad Q_{ij} h_{ij} \frac{\psi^4}{v e^\phi} \leq 0.$$

Hence,

$$(3.7) \quad w_l w_l = -v \phi'(s) \left(\frac{w}{\psi^2} \right)_i,$$

and

$$(3.8) \quad \frac{\psi^4}{v} Q_{ij} w_{lij} w_l + (\phi''(s) - (\phi'(s))^2) Q_{ij} \left(\frac{w}{\psi^2} \right)_i \left(\frac{w}{\psi^2} \right)_j \psi^4 + \phi'(s) Q_{ij} \left(\frac{w}{\psi^2} \right)_{ij} \psi^4 \leq 0.$$

By (3.1), we have

$$\begin{aligned} \psi^4 \partial_i \left(\frac{w}{\psi^2} \right) \partial_j \left(\frac{w}{\psi^2} \right) &= w_i w_j + O(|\nabla w| \psi) + \frac{4w^2 \psi_i \psi_j}{\psi^2}, \\ \psi^4 \partial_{ij} \left(\frac{w}{\psi^2} \right) &= w_{ij} \psi^2 + O(|\nabla w| \psi + \psi^2). \end{aligned}$$

We will prove later $\phi''(s) - (\phi'(s))^2 > 0$. Then, (3.8) reduces to

$$(3.9) \quad \begin{aligned} 0 &\geq \frac{1}{v} Q_{ij} w_{lij} w_l \psi^4 + (\phi''(s) - (\phi'(s))^2) Q_{ij} (w_i w_j + O(|\nabla w| \psi + 1)) \\ &\quad + \phi'(s) Q_{ij} (w_{ij} \psi^2 + O(|\nabla w| \psi + 1)). \end{aligned}$$

By the properties in (3.6), we have

$$(3.10) \quad \begin{aligned} Q_{ij} (w_{ij} \psi^2) &= T_{ij} (\psi^2 \bar{A} (w - \log \psi + \tilde{c}_1 \psi)_{ij} - \psi^2 |\nabla w|^2 \delta_{ij} \\ &\quad + \psi^2 \partial_i w \partial_j w + O(|\nabla w| \psi + 1)) \\ &= k \beta_{n,k} e^{2k \tilde{c}_1 \psi} e^{2kw} \\ &\quad + T_{ij} (-\psi^2 |\nabla w|^2 \delta_{ij} + \psi^2 \partial_i w \partial_j w + O(|\nabla w| \psi + 1)). \end{aligned}$$

Next, by applying ∂_m to (3.4), we obtain

$$(3.11) \quad \begin{aligned} &T_{ij} \left(2\psi \partial_m \psi (\partial_{ij} w + \frac{1}{n-2} \Delta w \delta_{ij} + |\nabla w|^2 \delta_{ij} - \partial_i w \partial_j w) \right. \\ &\quad + \psi^2 (\partial_{ijm} w + \frac{1}{n-2} \Delta w_m \delta_{ij} - 2v \phi'(s) \partial_m \left(\frac{w}{\psi^2} \right) \delta_{ij} - \partial_i w \partial_{jm} w - \partial_{im} w \partial_j w) \\ &\quad \left. - 2\partial_{lm} w (\partial_l \psi \psi - \partial_l (\tilde{c}_1 \psi) \psi^2) \delta_{ij} + 2\partial_{jm} w (\partial_i \psi \psi - \partial_i (\tilde{c}_1 \psi) \psi^2) + O(1 + |\nabla w|) \right) \\ &= 2k \beta_{n,k} e^{2k \tilde{c}_1 \psi} e^{2kw} \partial_m w + 2k \beta_{n,k} e^{2k \tilde{c}_1 \psi} e^{2kw} \partial_m (\tilde{c}_1 \psi). \end{aligned}$$

We multiply (3.11) by $\frac{1}{v} \psi^2 \partial_m w$ and sum over m . Then by (3.6) and (3.7), we get

$$(3.12) \quad \begin{aligned} \frac{1}{v} Q_{ij} w_{lij} w_l \psi^4 &= \frac{2}{v} k \beta_{n,k} e^{2k \tilde{c}_1 \psi} e^{2kw} |\nabla w|^2 \psi^2 + O(1) \\ &\quad + T_{ij} (2\phi'(s) \psi^2 |\nabla w|^2 \delta_{ij} - 2\phi'(s) \psi^2 w_i w_j + O(1 + |\nabla w| \phi'(s))). \end{aligned}$$

Note $0 < \phi'(\frac{w}{\psi^2}), \phi''(\frac{w}{\psi^2}) < 1$ and substitute (3.10), (3.12) and

$$Q_{ij}w_iw_j = T_{ij}w_iw_j + \frac{1}{n-2}T_{ll}|\nabla w|^2$$

into (3.9). Then, we have

$$\begin{aligned} 0 &\geq O(1) + T_{ij}((\phi''(s) - (\phi'(s))^2)w_iw_j + (\phi''(s) - (\phi'(s))^2)\frac{1}{n-2}|\nabla w|^2\delta_{ij} \\ &\quad + 2\phi'(s)\psi^2|\nabla w|^2\delta_{ij} - 2\phi'(s)\psi^2w_iw_j - \phi'(s)\psi^2|\nabla w|^2\delta_{ij} + \phi'(s)\psi^2w_iw_j \\ &\quad + O(1 + |\nabla w|)) \\ &= O(1) + T_{ij}((\phi''(s) - (\phi'(s))^2 - \phi'(s)\psi^2)w_iw_j \\ &\quad + (\frac{1}{n-2}\phi''(s) - \frac{1}{n-2}(\phi'(s))^2 + \phi'(s)\psi^2)|\nabla w|^2\delta_{ij} + O(1 + |\nabla w|)) \end{aligned}$$

By the expression of ϕ , we have, for a large constant C ,

$$\phi'(\frac{w}{\psi^2}) > \frac{1}{p3^pC_0},$$

and

$$\phi''(\frac{w}{\psi^2}) - (\phi')^2(\frac{w}{\psi^2}) - C\phi'(\frac{w}{\psi^2}) > \frac{1}{p3^pC_0^2}(p-1-\frac{1}{p}-3CC_0).$$

Fix p large enough. Then, we have, for some positive ϵ ,

$$\begin{aligned} C &\geq \epsilon T_{ij}w_iw_j + T_{ij}(2\epsilon|\nabla w|^2\delta_{ij} + O(1 + |\nabla w|)) \\ &\geq \epsilon T_{ij}w_iw_j + T_{ij}(\epsilon|\nabla w|^2\delta_{ij} + O(1)\delta_{ij}), \end{aligned}$$

where we used the fact $|T_{ij}|^2 \leq T_{ii}T_{jj}$. Take B large to be determined.

Case 1. The matrix $\epsilon|\nabla w|^2\delta_{ij} + O(1)\delta_{ij}$ has an eigenvalue less than B . In this case, the gradient estimate is immediate.

Case 2. The matrix $\epsilon|\nabla w|^2\delta_{ij} + O(1)\delta_{ij}$ has all eigenvalues bigger than B . By absorbing lower order terms, we have

$$C \geq \epsilon T_{ij}w_iw_j + T_{ll}B.$$

By (3.6), we have $\sigma_{k-1} \leq C$, independent of B . Then by Proposition 4.2 in [6], (3.6) and the positive lower bound for σ_k , we can fix B large enough to get a contradiction.

Then, we have $|\nabla w|^2(x_0) \leq C$. This finishes the proof. \square

We now improve Lemma 3.1 under the same assumption.

Proof of Theorem 1.3. Take α as in Theorem 1.3, ψ as in the proof of Lemma 3.1 and $\tilde{c}_1, \dots, \tilde{c}_{n,1} \in C^\infty(\Omega)$ satisfying

$$\tilde{c}_1 = c_1, \dots, \tilde{c}_{n,1} = c_{n,1} \quad \text{in } D_{\delta_2/2},$$

where $c_i, i = 1, \dots, n-1$, and $c_{n,1}$ are functions as in (2.6), (2.7) and (2.8) and we rewrite the constant δ_3 in Lemma 3.1 as δ_2 .

Set

$$(3.13) \quad f = \tilde{c}_1 \psi + \cdots + \tilde{c}_{n,1} \psi^n \log \psi,$$

and

$$w = u + \log \psi - f.$$

First, we will prove, for some $C_0 > 1$,

$$(3.14) \quad \left| \frac{w}{\psi^{2+2\alpha}} \right| \leq C_0 \quad \text{in } \Omega.$$

By Theorem 1.1, (3.14) holds in $D_{\delta_2/2}$. We point out that, in order to apply Theorem 1.1, we require $2 + 2\alpha \leq n$, which results in the choice of α in the statement of Theorem 1.3. Next, using u_{j_1} and u_{j_2} , obtained in the proof of Lemma 3.1, we know that (3.14) holds in $\Omega \setminus D_{\delta_2/2}$.

We rewrite the equation (1.5) as

$$(3.15) \quad \sigma_k(\psi^2(\overline{A}(w - \log \psi + f))) = e^{2kf} \left(\frac{n-1}{n-2} \right)^k C_n^k e^{2kw} \doteq e^{2kf} \beta_{n,k} e^{2kw} \quad \text{in } \Omega,$$

where $(\overline{A}(u))_{ij}$ is as in (3.5) and f is as in (3.13). We use $T_{k-1} \doteq T$ for $(k-1)$ -Newton transformation associated with $\psi^2 \overline{A}(w - \log \psi + f)$, which is positive since $\psi^2 \overline{A} \in \Gamma_k^+$. Set

$$Q_{ij} = T_{ij} + \frac{1}{n-2} T_{ll} \delta_{ij}.$$

Then, Q_{ij} is positive definite by [6]. By the properties in (3.6), we have

$$\begin{aligned} Q_{ij}(w_{ij} \psi^2) &= T_{ij}(\psi^2 \overline{A}(w - \log \psi + f)_{ij} - \psi^2 |\nabla w|^2 \delta_{ij} \\ &\quad + \psi^2 \partial_i w \partial_j w + O(|\nabla w| \psi + 1)) \\ &= k \beta_{n,k} e^{2kf} e^{2kw} + T_{ij}(-\psi^2 |\nabla w|^2 \delta_{ij} + \psi^2 \partial_i w \partial_j w + O(|\nabla w| \psi + 1)), \end{aligned}$$

and hence

$$(3.16) \quad Q_{ij} w_{ij} > T_{ij}(-|\nabla w|^2 \delta_{ij} + \partial_i w \partial_j w + O(\frac{|\nabla w|}{\psi} + \frac{1}{\psi^2})).$$

Set

$$\phi(s) = \frac{1}{p^2(3C_0)^p} (2C_0 + s)^p,$$

for some p large to be determined and C_0 in (3.14). Then,

$$\frac{1}{p^2} \geq \phi(s) > 0 \quad \text{for any } s \in [-C_0, C_0].$$

Set

$$h = (1 + \frac{1}{2} |\nabla(\frac{w}{\psi^\alpha})|^2) e^{\phi(\frac{w}{\psi^{2+2\alpha}})} \doteq v e^{\phi(\frac{w}{\psi^{2+2\alpha}})}.$$

We will prove, for some constant C ,

$$|h|_{L^\infty(\Omega)} \leq C.$$

This would imply the desired conclusion.

First, for an arbitrary point $x_0 \in \partial\Omega$, we can argue similarly as in the proof of Lemma 3.1. Note that w defined in this proof satisfies $\frac{w}{\psi^\alpha} \equiv 0$ on $\partial\Omega$ and $\frac{w}{\psi^\alpha} \leq Cd^{2+\alpha}$ in D_{δ_2} . Then, $\nabla_{x'}(\frac{w}{\psi^\alpha}) \equiv 0$ on $\partial\Omega$ and

$$|\frac{\partial}{\partial\nu}(\frac{w}{\psi^\alpha})(x_0)| = |\lim_{d \rightarrow 0} \frac{(\frac{w}{\psi^\alpha})(x'_0, d) - 0}{d - 0}| = 0.$$

Hence, $\nabla(\frac{w}{\psi^\alpha})(x_0) = 0$, implying $|h(x_0)| \leq C$.

Thus, without loss of generality, we can assume that the maximum of h attains at a point $x_0 \in \Omega$. The proof is inspired by [6]. Take A large to be determined. Without of generality, we assume $|\nabla(\frac{w}{\psi^\alpha})(x_0)| \geq A$ is sufficiently large. Otherwise the conclusion is obvious. All calculation below is at x_0 . For brevity, we write

$$s = \frac{w}{\psi^{2+2\alpha}}.$$

By differentiating h once, we have $h_i = 0$ and hence

$$(3.17) \quad (\frac{w}{\psi^\alpha})_{li}(\frac{w}{\psi^\alpha})_l = -v\phi'(s)(\frac{w}{\psi^{2+2\alpha}})_i.$$

Using (3.14), we have

$$\begin{aligned} (\frac{w}{\psi^\alpha})_i &= \frac{w_i}{\psi^\alpha} + O(\psi^{\alpha+1}), \\ \partial_{ij}(\frac{w}{\psi^\alpha}) &= \frac{w_{ij}}{\psi^\alpha} + O(\frac{|\nabla w|}{\psi^{\alpha+1}}), \\ \partial_{ij}(\frac{w}{\psi^{2\alpha+2}}) &= \frac{w_{ij}}{\psi^{2\alpha+2}} + O(\frac{|\nabla w|}{\psi^{2\alpha+3}} + \frac{1}{\psi^2}). \end{aligned}$$

Apply ∂_m to (3.15) and then by Lemma 3.1, we have

$$\begin{aligned} (3.18) \quad Q_{ij}(\frac{w}{\psi^\alpha})_{ijm} &= T_{ij}\{-2(\frac{w}{\psi^\alpha})_{lm}(\frac{w}{\psi^\alpha})_l\psi^\alpha\delta_{ij} + (\frac{w}{\psi^\alpha})_{im}(\frac{w}{\psi^\alpha})_j\psi^\alpha + (\frac{w}{\psi^\alpha})_{jm}(\frac{w}{\psi^\alpha})_i\psi^\alpha \\ &\quad + (\frac{w}{\psi^\alpha})_{im}O(\frac{1}{\psi}) + O(|\nabla(\frac{w}{\psi^\alpha})|\frac{1}{\psi^2} + \frac{1}{\psi^{3+\alpha}} + |\nabla(\frac{w}{\psi^\alpha})|^2\frac{1}{\psi^{1-\alpha}})\} \\ &\quad + Q_{ij}w_{ij}O(\frac{1}{\psi^{1+\alpha}}) + O(\frac{1}{\psi^{3+\alpha}} + |\nabla(\frac{w}{\psi^\alpha})|\frac{1}{\psi^2}). \end{aligned}$$

Next, differentiate h one more time. Since Q_{ij} is positive definite, we have $0 \geq Q_{ij}h_{ij}\frac{1}{ve^\phi}$ and hence

$$\begin{aligned} (3.19) \quad 0 &\geq \frac{1}{v}Q_{ij}(\frac{w}{\psi^\alpha})_{lij}(\frac{w}{\psi^\alpha})_l + (\phi''(s) - (\phi'(s))^2)Q_{ij}(\frac{w}{\psi^{2+2\alpha}})_i(\frac{w}{\psi^{2+2\alpha}})_j \\ &\quad + \phi'(s)Q_{ij}(\frac{w}{\psi^{2+2\alpha}})_{ij}. \end{aligned}$$

We sum (3.18) with $\frac{1}{v}(\frac{w}{\psi^\alpha})_m$. Note $0 < \phi'(\frac{w}{\psi^2}), \phi''(\frac{w}{\psi^2}) < 1$. Then,

$$\begin{aligned}
 & \frac{1}{v} Q_{ij}(\frac{w}{\psi^\alpha})_{ijm}(\frac{w}{\psi^\alpha})_m \\
 &= T_{ij} \{ 2\phi'(s)\psi^{2+2\alpha}(|\nabla(\frac{w}{\psi^{2+2\alpha}})|^2\delta_{ij} - (\frac{w}{\psi^{2+2\alpha}})_i(\frac{w}{\psi^{2+2\alpha}})_j) \\
 &+ \phi'(s)|\nabla(\frac{w}{\psi^{2+2\alpha}})|O(\frac{1}{\psi}) + O(\frac{1}{\psi^2} + \frac{1}{\psi^{3+\alpha}}\frac{1}{A} + |\nabla(\frac{w}{\psi^\alpha})|\frac{1}{\psi^{1-\alpha}}) \} \\
 &+ Q_{ij}w_{ij}O(\frac{1}{\psi^{1+\alpha}}\frac{1}{A}) + O(\frac{1}{\psi^{3+\alpha}}\frac{1}{A} + \frac{1}{\psi^2}).
 \end{aligned} \tag{3.20}$$

Note $\phi'(\frac{w}{\psi^2}) > \frac{1}{p3^pC_0}$ and we will prove later $\phi''(s) - (\phi'(s))^2 > 0$. Then by (3.16), (3.20) and Lemma 3.1, (3.19) reduces

$$\begin{aligned}
 0 &\geq O(\frac{1}{\psi^{3+\alpha}}\frac{1}{A} + \frac{1}{\psi^2}) \\
 &+ T_{ij} \left\{ ((\phi''(s) - (\phi'(s))^2)\frac{1}{n-2} + 2\phi'(s)\psi^{2+2\alpha})|\nabla(\frac{w}{\psi^{2+2\alpha}})|^2\delta_{ij} \right. \\
 &+ ((\phi''(s) - (\phi'(s))^2) - 2\phi'(s)\psi^{2+2\alpha})(\frac{w}{\psi^{2+2\alpha}})_i(\frac{w}{\psi^{2+2\alpha}})_j \\
 &\left. + \phi'(s)|\nabla(\frac{w}{\psi^{2+2\alpha}})|O(\frac{1}{\psi}) + O(\frac{1}{\psi^{4+2\alpha}}) \right\}.
 \end{aligned} \tag{3.21}$$

Multiply (3.21) by $\psi^{4+2\alpha}$. By

$$(\frac{w}{\psi^{2+2\alpha}})_i = (\frac{w}{\psi^\alpha})_i(\frac{1}{\psi^{2+\alpha}}) + O(\frac{1}{\psi}),$$

we have

$$\begin{aligned}
 0 &\geq O(1) + T_{ij} \left\{ ((\phi''(s) - (\phi'(s))^2)\frac{1}{n-2} + 2\phi'(s)\psi^\alpha)|\nabla(\frac{w}{\psi^\alpha})|^2\delta_{ij} \right. \\
 &+ ((\phi''(s) - (\phi'(s))^2) - 2\phi'(s)\psi^\alpha)(\frac{w}{\psi^\alpha})_i(\frac{w}{\psi^\alpha})_j \\
 &\left. + |\nabla(\frac{w}{\psi^\alpha})|O(1) + O(1) \right\}.
 \end{aligned} \tag{3.22}$$

By the expression of ϕ , for a large constant C , we have

$$\begin{aligned}
 \phi'(\frac{w}{\psi^2}) &> \frac{1}{p3^pC_0}, \\
 \phi''(\frac{w}{\psi^2}) - (\phi')^2(\frac{w}{\psi^2}) - C\phi'(\frac{w}{\psi^2}) &> \frac{1}{p3^pC_0^2}(p-1 - \frac{1}{p} - 3CC_0).
 \end{aligned}$$

Fix p large enough. Then, we have, for some positive ϵ ,

$$C \geq \epsilon T_{ij}(\frac{w}{\psi^\alpha})_i(\frac{w}{\psi^\alpha})_j + T_{ij}(2\epsilon|\nabla(\frac{w}{\psi^\alpha})|^2 + O(1))\delta_{ij},$$

where we used the fact $|T_{ij}|^2 \leq T_{ii}T_{jj}$. Take B large to be determined and we consider two cases.

Case 1. If the matrix

$$2\epsilon|\nabla(\frac{w}{\psi^\alpha})|^2\delta_{ij} + O(1)\delta_{ij}$$

has an eigenvalue less than B , then the gradient estimate is immediate.

Case 2. Otherwise, absorbing lower order terms, we have

$$C \geq \epsilon T_{ij}(\frac{w}{\psi^\alpha})_i(\frac{w}{\psi^\alpha})_j + BT_{ll}.$$

We argue similarly as in the proof of Lemma 3.1. Then we have $|\nabla(\frac{w}{\psi^\alpha})|(x_0) \leq C$. \square

Remark 3.2. We emphasize again that the validity of (3.14) requires a relation of α and n . In fact, for a general $\alpha \geq \frac{1}{2}$ and w defined above, when $n \geq 2 + 2\alpha$, we have

$$|\frac{w}{\psi^{2+2\alpha}}| \leq C, \quad |\frac{\nabla w}{\psi^\alpha}| \leq C.$$

4. THE C^2 -ESTIMATES

In this section, we derive estimates of second derivatives.

Proof of Theorem 1.4. Take w , ψ and f as defined in the proof of Theorem 1.3. This proof is divided into two steps.

Step 1. We will prove that there exists a constant C , depending only on $\partial\Omega$, n and k , such that

$$\Delta w \geq -C \quad \text{in } \Omega.$$

We proceed to prove this in $D_{\delta_3/2}$, where δ_3 is the constant in Theorem 1.3. The proof in $\Omega \setminus D_{\delta_3/2}$ is similar but easier.

By (1.5)-(1.6), Theorem 1.1 and noting $\psi = d$ in $D_{\frac{1}{2}\delta_3}$, we have, in $D_{\frac{1}{2}\delta_3}$,

$$\begin{aligned} (4.1) \quad \frac{n}{(C_n^k)^{1/k}} \left(\sigma_k \left(\lambda \left(\frac{1}{n-2} A(u) \right) \right) \right)^{1/k} &= \frac{1}{d^2} \left(\frac{n(n-1)}{n-2} \right) e^{2(w+f)} \\ &= \frac{1}{d^2} \left(\frac{n(n-1)}{n-2} \right) (1 + 2c_1 d + O(d^2)). \end{aligned}$$

By the expression of $A(u)$ in (1.4), a straightforward calculation yields, in $D_{\frac{1}{2}\delta_3}$,

$$\begin{aligned} (4.2) \quad \sigma_1 \left(\lambda \left(\frac{1}{n-2} A(u) \right) \right) &= \left(1 + \frac{n}{n-2} \right) \Delta u + (n-1) |\nabla u|^2 \\ &= \left(1 + \frac{n}{n-2} \right) \Delta w + \frac{1}{d^2} \left(\frac{n(n-1)}{n-2} \right) + \frac{c_1}{d} \frac{2n(n-1)}{n-2} + O(1), \end{aligned}$$

where we used the fact that $-\Delta d = H_{\partial\Omega} + O(d)$ and the definition of c_1 in (2.6). By Maclaurin's inequality, we have

$$\sigma_1 \left(\lambda \left(\frac{1}{n-2} A(u) \right) \right) \geq \frac{n}{(C_n^k)^{1/k}} \sigma_k^{1/k} \left(\lambda \left(\frac{1}{n-2} A(u) \right) \right).$$

By combining with (4.1) and (4.2) and by a straightforward calculation, we have $\Delta w \geq -C$.

Step 2. Next, we will prove

$$\max_{\gamma \in \mathbb{S}^{n-1}, p \in \overline{\Omega}} \partial_\gamma \partial_\gamma w \leq C,$$

where C is a positive constant depending only on Ω , n and k . The proof of this step is inspired by [6].

First, assume $n \geq 2 + 2\alpha$, for some $\alpha \geq 1$. Later on, we will take $\alpha = 3$ but we write it in the present form to demonstrate why we choose $\alpha = 3$. By Remark 3.2, we have

$$(4.3) \quad \left| \frac{w}{\psi^{2+2\alpha}} \right| \leq C, \quad \left| \frac{\nabla w}{\psi^\alpha} \right| \leq C.$$

Hence, $\nabla w \equiv 0$ on $\partial\Omega$. Moreover, in principal coordinates at any boundary point x_0 with e_n as the unit inner normal vector to $\partial\Omega$ at x_0 , we have

$$\nabla_{x'} \nabla w(x_0) = 0,$$

and

$$|\nabla_n \nabla w(x_0)| \leq \lim_{d \rightarrow 0} \left| \frac{Cd - 0}{d} \right| = C.$$

Therefore, we obtain

$$(4.4) \quad |\nabla^2 w|_{L^\infty(\partial\Omega)} \leq C.$$

Next, set

$$(4.5) \quad h(p, \gamma) = \partial_\gamma \partial_\gamma w(p) + \Lambda \frac{|\nabla w|^2}{\psi^{2\alpha}}(p) \quad \text{for } (p, \gamma) \in \overline{\Omega} \times \mathbb{S}^{n-1},$$

where Λ is a constant to be determined. We will prove

$$|h|_{L^\infty(\overline{\Omega} \times \mathbb{S}^{n-1})} \leq C,$$

which implies the conclusion in Step 2 by (4.3).

Without loss of generality, we assume that the maximum of h attains at $(p, \gamma) \in \Omega \times \mathbb{S}^{n-1}$. Otherwise, by (4.4), the conclusion is immediate. Then, by rotating coordinates at p , we may assume $\frac{\partial}{\partial x_1} = \gamma$. Set

$$\tilde{h}(x) = h(x, \frac{\partial}{\partial x_1}) = w_{11} + \Lambda \frac{|\nabla w|^2}{\psi^{2\alpha}}.$$

Without loss of generality, we can assume $w_{11}(p) \geq 1$. Otherwise, the desired result is immediate. Since p is the maximum point of \tilde{h} , we have, at p ,

$$(4.6) \quad 0 = \partial_i \tilde{h} = w_{11i} + \Lambda \frac{2w_k w_{ki}}{\psi^{2\alpha}} - 2\alpha \Lambda \frac{\psi_i |\nabla w|^2}{\psi^{2\alpha+1}},$$

and

$$(4.7) \quad \begin{aligned} 0 \geq \partial_{ij} \tilde{h} &= w_{11ij} + \Lambda \frac{2w_{kj}w_{ki}}{\psi^{2\alpha}} - 4\alpha\Lambda \frac{w_{kj}w_k\psi_i}{\psi^{2\alpha+1}} - 4\alpha\Lambda \frac{w_{ki}w_k\psi_j}{\psi^{2\alpha+1}} \\ &\quad - 2\alpha\Lambda \frac{\psi_{ij}|\nabla w|^2}{\psi^{2\alpha+1}} + 2\alpha(2\alpha+1)\Lambda \frac{\psi_i\psi_j|\nabla w|^2}{\psi^{2\alpha+2}} + \Lambda \frac{2w_k w_{kij}}{\psi^{2\alpha}}. \end{aligned}$$

All the calculation below is at p . Recall from Section 3 that

$$Q_{ij} = T_{ij} + \frac{1}{n-2} T_{ll} \delta_{ij}.$$

Since Q_{ij} is positive definite, using (4.3) and (4.7), we have, at p ,

$$(4.8) \quad 0 \geq Q_{ij} w_{11ij} + \Lambda \frac{2}{\psi^{2\alpha}} Q_{ij} w_{kj} w_{ki} + \Lambda O\left(\frac{1}{\psi^\alpha}\right) Q_{ij} w_{kij} + \Lambda T_{ij} \delta_{ij} O\left(\frac{1}{\psi^2} + \frac{|\nabla^2 w|}{\psi^{\alpha+1}}\right).$$

We write $\bar{A}_{ij} = \bar{A}_{ij}(u)$ for convenience.

First, we consider the term $Q_{ij} w_{kij}$. By (3.6) and (4.3), differentiating (3.15) with respect to x_m , we have

$$\begin{aligned} \partial_m(\text{L.H.S.}) &= \partial_m \sigma_k(\psi^2 \bar{A}_{ij}) = T_{ij}(2\psi \psi_m \bar{A}_{ij} + \psi^2 (\bar{A}_{ij})_m) = \psi^2 T_{ij} (\bar{A}_{ij})_m + k \sigma_k \frac{2\psi_m}{\psi}, \\ \partial_m(\text{R.H.S.}) &= \partial_m(e^{2k(f+w)} \beta_{n,k}) = O(1). \end{aligned}$$

Hence,

$$(4.9) \quad T_{ij} (\bar{A}_{ij})_m = O\left(\frac{1}{\psi^3}\right).$$

On the other hand, substituting $u = -\log \psi + w + f$ in $\bar{A}_{ij}(u)$ and then by (4.3), we have

$$(4.10) \quad Q_{ij} w_{ijm} = O\left(\frac{1}{\psi^3}\right)(1 + T_{ij} \delta_{ij}) + O\left(\frac{|\nabla^2 w|}{\psi}\right) T_{ij} \delta_{ij}.$$

Next, we consider the term $Q_{ij} w_{11ij}$. Set $\sigma = (\sigma_k)^{1/k}$. Then

$$(4.11) \quad \sigma(\lambda(\psi^2 \bar{A}_{ij})) = e^{2(f+w)} \beta_{n,k}^{1/k}.$$

Differentiate (4.11) twice with respect to x_1 and compare the R.H.S. with the L.H.S. By (4.3) and the concavity of σ , we have

$$(4.12) \quad T_{ij}(\psi^2 \bar{A}_{ij})_{11} \geq -C - C w_{11}.$$

Substituting $u = -\log \psi + w + f$ in $\bar{A}_{ij}(u)$ and then by (3.6), (4.3), (4.6) and (4.9), we have

$$(4.13) \quad Q_{ij} w_{11ij} \geq -\frac{C}{\psi^2} - \frac{C}{\psi^2} w_{11} + T_{ij} \delta_{ij} O\left(\frac{1}{\psi^4} + |\nabla^2 w|^2 + \frac{|\nabla^2 w|}{\psi^{\alpha+1}}\right).$$

Substitute (4.10) and (4.13) into (4.8). Then, multiply (4.8) by $\psi^{3+\alpha}$. By (4.3), we have

$$\begin{aligned} 0 \geq & -C - C\psi^{\alpha+1}w_{11} + T_{ij}\delta_{ij}\left(\frac{1}{\psi^{\alpha-3}}\frac{2\Lambda}{n-2}\sum_{i,k}|w_{ki}|^2 + O((1+\Lambda) + \psi^{\alpha+3}|\nabla^2 w|^2\right. \\ & \left. + (1+\Lambda)|\nabla^2 w|\psi^2) + \frac{2\Lambda}{\psi^{\alpha-3}}T_{ij}w_{kj}w_{ki}.\end{aligned}$$

Choose Λ large enough and, without loss of generality, we may assume $\sum_{i,k}|w_{ki}|^2$ is large and much larger than Λ . Then we have, for a positive constant c ,

$$(4.14) \quad C + C\psi^{\alpha+1}w_{11} \geq \sigma_{k-1}(\psi^2\bar{A}_{ij})c\sum_{i,k}|w_{ki}|^2 \geq \sigma_{k-1}(\psi^2\bar{A}_{ij})cw_{11}^2.$$

On the other hand, by Maclaurin's inequality, we have

$$\sigma_{k-1} \geq \left(\frac{\sigma_k}{\binom{n}{k}}\right)^{\frac{k-1}{k}} \binom{n}{k-1}.$$

Note that, for some positive c_0 ,

$$\sigma_k(\psi^2\bar{A}_{ij}) = e^{2k(f+w)}\beta_{n,k} > c_0,$$

where we used (4.3) and the definition of f . Then, we have, for some positive c_1 ,

$$\sigma_{k-1}(\psi^2\bar{A}_{ij}) > c_1.$$

Hence, (4.14) implies, for some positive constant ϵ_0 ,

$$C + Cw_{11} \geq \epsilon_0w_{11}^2.$$

Then, we draw the conclusion $w_{11} \leq C$ and finish the proof in Step 2.

Combining the two steps, we have the desired conclusion. \square

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